ENERGY OF FORMATION OF A TORNADOLIKE VORTEX

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The energetic conditions of formation of a tornadolike vortex are studied theoretically and experimentally.

1. A water model of a tornadolike vortex is created in the following way (Fig. 1). The shaft of an electric motor passes through the bottom of a rectangular vessel, the inside dimensions of which are 12.5 × 12.5 × 20 cm. A metal disk 6 cm in diameter is fastened to the shaft. Inside the rectangular vessel there is a hollow glass cylinder (inside diameter 8 cm) isolating the disk from the corner zones of the vessel. Another thin metal disk with openings along the perimeter is connected to the shaft of the electric motor by a belt drive. There were 30 openings, which was determined by the principle of operation of the angular-velocity meter and the chosen transmission ratio of the belt drive. A TsAT-2M automatic digital tachometer, the photoelectric pickup of which is brought up to the disk with openings, is used to measure the angular velocity. The side surface of the rectangular vessel is illuminated by a light beam from outside on the left, so that the opposite surface serves as a screen.

The electric motor is connected to the power grid through a rectifier and two transformers, which permits smooth variation of the voltage on the motor terminals, measured during a test.

The glass vessel does not lie flush against the bottom of the rectangular vessel, so that the same water level $h_0$ is established in the entire system at the start of a test. Then as the disk rotates, rotational motion develops in the adjacent layer of liquid, which causes a pressure increase toward the periphery of the disk. The pressure nonuniformity over the radius of the disk results in a pressure drop along the height of the liquid column. The latter promotes the development of a vortex funnel, the length of which increases as the motor

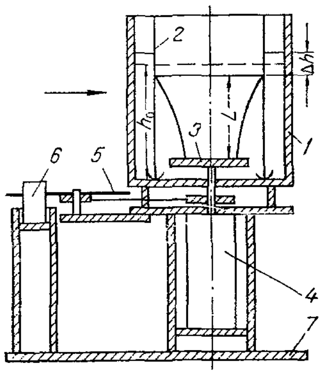


Fig. 1. Schematic plan of the experimental apparatus: 1) transparent rectangular vessel; 2) hollow glass cylinder; 3) metal disk; 4) electric motor; 5) disk with openings along the perimeter; 6) photoelectric pickup; 7) massive metal base.

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speed increases. At a certain motor speed $\omega_1$ the funnel touches the disk and a tornado is formed with a certain length L not coinciding with the level $h_0$. The observations show that the funnel, tornado, and disk represent an autooscillatory system, since with a constant voltage on the motor terminals the funnel undergoes oscillatory motions along the vertical axis, which in turn results in variation of the motor speed. The indicated character of the interaction of the funnel with the disk results in the fact that at certain motor speeds close to $\omega_1$, the funnel, in undergoing oscillatory motions, can touch the disk and separate from it after a certain time, i.e., the tornado which formed did not "live" long. To prolong the "lifetime" of the tornado one must increase the voltage on the motor terminals. Then a tornado of the same length will have a greater area in contact with the disk. The latter requires the introduction of an additional parameter $r_0$, equal to the radius of the lower base of the tornado in contact with the disk.

The contours of the vortex funnel and the tornado are well seen through the glass cylinder and the walls of the rectangular vessel. This makes it possible to draw the line of the free surface of the tornado on tracing paper glued to the screen. In projecting the tornado onto the screen its geometrical dimensions are distorted. But the values of $H_0$ and the disk radius R allow one to convert these tornado dimensions to the true ones.

The observations show that with this means of exciting vortical motion there is descending axial flow inside the tornado and ascending flow in the surrounding medium. The descending flow dominates over the ascending flow, however, because of which an excess pressure is created near the disk at the bottom of the rectangular vessel. The hollow glass cylinder does not lie flush against the bottom of the vessel, so that liquid flows into the space between the vessel and the cylinder under the action of this excess pressure. As a result of this, the water level becomes higher than the initial level $h_0$, while the upper surface of the tornado drops below $h_0$. A difference in levels $\Delta h$ develops (Fig. 1), which determines the pressure of the tornado against the bottom of the vessel.

The measurements were made in tornadoes for which $r_0 = R/2$.

The essence of a test consisted in the following. The water level $h_0$ is assigned. A tornado with a certain length L and the indicated value of $r_0$ is excited by smooth variation of the voltage on the motor terminals. The contour of the tornado is sketched on the tracing paper glued to the screen and the readings of the automatic digital tachometer are recorded simultaneously. After the tornado contours were fixed, the increase in voltage on the motor terminals was stopped, but despite this, the position of the free surface varies near the initially fixed contour. The tachometer readings, which have an oscillatory character, also vary in accordance with this.

The duration of a test is determined by the time during which the position $r_0$ on the screen varies by 1-2 mm in one direction or the other. Then the voltage on the motor terminals is smoothly decreased. The tornado narrows, and at some lower speed it is torn off from the disk and changes into a vortex funnel.

Since the tachometer readings were recorded by hand, the test was run three times for a given $h_0$. After this the average speed $\omega_1$ at which a tornado with a given length L and radius $r_0$ developed and the corresponding $\omega_2$ at which it disappeared were calculated. The average position of the free surface of the tornado was inscribed on the tracing paper and all the geometrical dimensions of the vortex were subsequently taken from it. Voltmeter readings, the repeatability of which essentially influences the repeatability of the energetic and geometrical parameters of the tornado, were taken in the course of a test.

According to the certification data of the TsAT-2M instrument, the accuracy in measuring motor speeds is 1.005 $\min^{-1}$. If one neglects the optical distortion of rays connected with the difference in media, the inaccuracy in measuring the geometrical parameters of a tornado does not exceed 1 mm.

During one series of tests the quantity $h_0$ was changed 11 times. The main data on two test series are presented in Table 1. These series illustrate the repeatability and the small scatter of the measurement results. The measured quantities were averaged over the results of these test series, and functions $\omega_1(L)$ and $\omega_2(L)$ were obtained as a result (Fig. 2). It was found that for one length L there are two different values of $\omega_1$ and $\omega_2$, i.e., the energy expended on forming a tornado does not equal the energy of disappearance of the vortex. It is seen from Table 1 that the value of $|\omega_1-\omega_2|$ considerably exceeds the scatter of the experimental data.

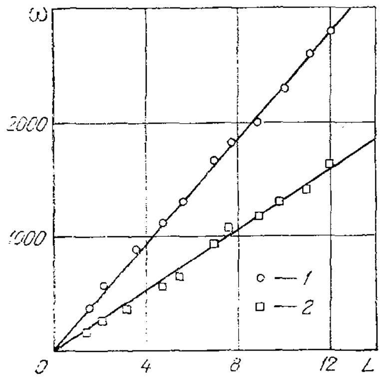



Fig. 2

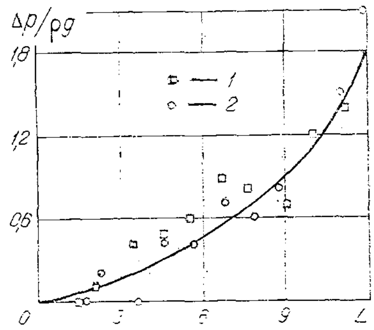



Fig. 3

Fig. 2. Disk speed as a function of vortex length: 1) speed at the development of a stable vortex; 2) speed at the disappearance of the vortex; ω, rpm; L, cm.

Fig. 3. Pressure of the tornado on the vessel bottom: 1) data of series I; 2) data of series II; Δp/ρg, cm.

The pressure exerted by the tornado on the bottom of the vessel depends nonlinearly on its length. The excess of this pressure over atmospheric as a function of the tornado length is presented in Fig. 3.

2. Let us analyze this tornadolike vortex from the standpoint of the theoretical model of a Rankine vortex. Then the tornado contour represents a free surface at which the pressure stays constant. The flow beyond the tornado is assumed to be potential, and the Bernoulli integral yields the following equation for the free surface:

$$z_1 = -\frac{A^2}{2gr^2}; \tag{1}$$

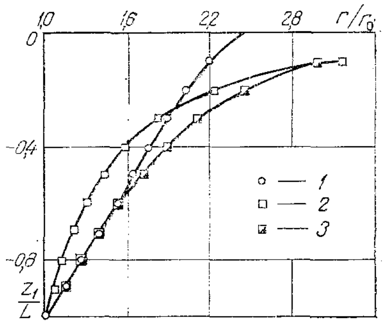
in addition, we designate

TABLE 1. Results of Measurements of Two Test Series

| Series I | | | | | Series II | | | | |
|---|---|---|---|---|---|---|---|---|---|
| L, cm | $U_1$, V | $\omega_1$, rpm | $U_2$, V | $\omega_2$, rpm | L, cm | $U_1$, V | $\omega_1$, rpm | $U_2$, V | $\omega_2$, rpm |
| 1,4 | 4,1 | 359 | 3,3 | 152 | 1,7 | 4,2 | 357 | 3,1 | 144 |
| 2,3 | 4,5 | 563 | 3,4 | 287 | 2,4 | 4,8 | 646 | 3,3 | 234 |
| 3,4 | 5,3 | 934 | 3,1 | 362 | 3,6 | 5,9 | 896 | 4,3 | 383 |
| 4,5 | 6,1 | 1162 | 4,0 | 528 | 4,5 | 6,5 | 1102 | 4,4 | 515 |
| 5,5 | 6,3 | 1311 | 4,1 | 664 | 5,6 | 6,2 | 1294 | 3,9 | 605 |
| 6,7 | 7,6 | 1654 | 4,8 | 919 | 6,8 | 8,7 | 1694 | 5,6 | 1035 |
| 7,6 | 9,5 | 1808 | 6,4 | 1093 | 7,9 | 9,5 | 1922 | 6,5 | 1048 |
| 9,0 | 9,7 | 1934 | 6,5 | 1206 | 8,8 | 9 | 1981 | 6 | 1195 |
| 10,0 | 10,7 | 2188 | 6,7 | 1371 | 10,0 | 11,2 | 2417 | 6,4 | 1411 |
| 11,2 | 12,3 | 2518 | 6,6 | 1416 | 11,1 | 14,1 | 2670 | 7,7 | 1449 |
| 12,2 | 13,5 | 2727 | 7,3 | 1722 | 11,9 | 15 | 2853 | 7,9 | 1724 |

TABLE 2. Results of a Calculation of the Circular Velocity and the Vortex Energy

| $L$, cm | $r_0$, cm | $v_1$, cm/sec | $v_2$, cm/sec | $v_3$, cm/sec | $E\cdot 10$, J |
|---|---|---|---|---|---|
| 2,7 | 1,52 | 103 | 61 | 73 | 0,02 |
| 3,6 | 1,59 | 149 | 63 | 84 | 0,04 |
| 4,5 | 1,56 | 180 | 84 | 94 | 0,06 |
| 6,8 | 1,52 | 269 | 164 | 115 | 0,13 |
| 10 | 1,48 | 374 | 219 | 140 | 0,24 |



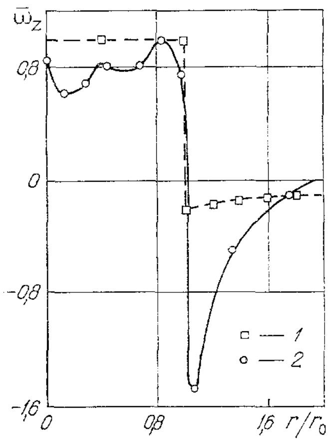


Fig. 4 Fig. 5

Fig. 4. Line of the free surface of the vortex: 1) experiment; 2) calculation from Eq. (4); 3) calculation from Eq. (7).

Fig. 5. Distribution of the curl of the velocity: 1) calculation from Eq. (9); 2) from observations in a real tornado.

$$z_1 = z - \frac{1}{g}\left(\text{const} - \frac{p}{\rho}\right). \tag{2}$$

Here for the tangential velocity we adopt the expression

$$v_3 = A/r. \tag{3}$$

The constant A is determined on the basis of the experimental values of $z_1 = -L$ and $r = r_0$. It proved to be different for each length L. The tangential velocity at the free surface of the vortex was determined from Eq. (3) and from the expressions $v_1 = \omega_1 r_0$ and $v_2 = \omega_2 r_0$. All three values of this velocity as a function of the tornado length are presented in Table 2.

The velocities $v_1$ and $v_3$ proved to be different. Consequently, the tangential velocity changes abruptly in the transition through the tornado boundary.

The kinetic energy of the potential flow is calculated as

$$E_0 = \int_0^1 \int_{r_0}^{R} \pi r v^2 \rho \, dr dz = \pi\rho A^2 \ln \frac{R}{r_0}.$$

From this it follows that $E_0 \to \infty$ as $R \to \infty$, i.e., the potential flow must have an infinite energy to maintain the tornado. This fact must be acknowledged as paradoxical.

To obtain a finite value of the kinetic energy of a tornado we use the equation

$$E = \int_0^1 \int_0^{r_0} \rho\pi\omega_1^2 r^3 dr dz = \frac{1}{4}\pi\rho\omega_1^2 r_0^4.$$

The function E(L) is shown in Table 2.

We compare Eq. (1) with experiment for a vortex with a length L = 10 cm. In this case $z_1 = -10$ cm and $r_0 = 1.48$ cm, while Eq. (1) is rewritten as

$$r = 4.6802/\sqrt{-z_1}. \tag{4}$$

From Fig. 4 it follows that Eq. (4) does not correspond to the experimental line of the free surface. The reason for the noncorrespondence must be sought in the incorrectness of Eq. (3) for the tangential velocity.

Let us assume that the circular velocity is defined as

$$v = \frac{A(z_1 + z_0)}{r^2}. \tag{5}$$

Then the equation of the free surface takes the form

$$z_1 = -\frac{A^2(z_1 + z_0)}{2gr^4}. \tag{6}$$

The constants A and $z_0$ are calculated from two test points: $z_1 = -10$ cm, r = 1.48 cm and $z_1 = -8$ cm, r = 1.86 cm. As a result, $z_0 = 14.84$ cm and A = 63.423 $cm^2$/sec. Now rewrite Eq. (6) as follows:

$$r^2 = (z_1 + 14.84)/0.6984\sqrt{-z_1}. \tag{7}$$

It is seen from Fig. 4 that near the lower boundary of the tornado there is a region where Eq. (7) coincides with the test curve. Therefore, near this region Eq. (5) is valid for the circular flow beyond the tornado.

The vortex under consideration corresponds to $\omega_1 = 253\ sec^{-1}$. Now the expression for the tangential velocity at $z_1 = -10$ cm can be defined as follows: Within the tornado the liquid adheres to the disk and rotates along with it as a rigid body, while beyond the tornado Eq. (5) holds, i.e.,

$$v = \omega_1 r, \quad r \leqslant r_0, \quad v = \frac{3.07 \cdot 10^2}{r^2}, \ r \geqslant r_0. \tag{8}$$

With allowance for Eq. (8) we can calculate the projection of the curl of the velocity onto the x axis. In the presence of axial symmetry

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial r} + \frac{v}{r}\right),$$

or finally,

$$\omega_z = \omega_1, \quad r \leqslant r_0, \quad 2\omega_z = -\frac{3.07 \cdot 10^2}{r^3}, \quad r \geqslant r_0. \tag{9}$$

It is seen from Eq. (9) that the quantity $\omega_z$ is constant within the tornado, and then at the free surface it abruptly falls to a negative value and then approaches zero.

The presence of a region with negative values of $\omega_z$ in a real tornado was first observed by Glaser [1]. The distribution of the dimensionless value of $\omega_z$ according to Eq. (9) and Glaser's calculations is shown in Fig. 5. It should be mentioned that the values of the radius on Glaser's curve refer to different heights, and it characterizes the behavior of $\omega_z$ in a tornado cross section only under the assumption that $\omega_z$ is constant vertically.

From Eqs. (8), (9) it follows that there exists a class of vortex flows in which there are large gradients of tangential velocity resulting in negative values of $\omega_z$. It is easy to show that $\omega_z < 0$ for all $n > 1$ if $v = 1/r^n$.

The question now arises of for what form of the hydrodynamic equations does v admit a solution of this type.

Let us consider the Reynolds equations for turbulent flows

$$\frac{\rho d\bar{v}_j}{dt} + \frac{\partial}{\partial x_k}(\rho\overline{v_j' v_k'}) = -\frac{\partial \bar{p}}{\partial x_j} + \frac{\partial}{\partial x_k}\bar{\tau}_{jk}, \tag{10}$$

where

$$\bar{\tau}_{jk} = \mu\left(\frac{\partial \bar{v}_j}{\partial x_k} + \frac{\partial \bar{v}_k}{\partial x_j}\right), \quad j, k = 1, 2, 3.$$

To close them the following hypothesis was proposed in [2]:

$$v_j' = b_0' \bar{v}_j + b_1'\left[(x_k - x_{0k})\frac{\partial \bar{v}_j}{\partial x_k}\right];$$

it reduces the above equations to the form

$$\frac{\partial\bar{\mathbf{V}}}{\partial t}+(1+\overline{b_0'^2}+\overline{b_0' b_1'})(\bar{\mathbf{V}}\cdot\nabla)\bar{\mathbf{V}}+\bar{\mathbf{V}}\left[\frac{\overline{b_0'^2}}{\rho}\operatorname{div}(\rho\bar{\mathbf{V}})+\overline{b_0' b_1'}\operatorname{div}\bar{\mathbf{V}}\right]=-\frac{1}{\rho}\operatorname{grad}\bar{p}+\nu\nabla^2\bar{\mathbf{V}}+\frac{1}{3}\nu\operatorname{grad}\operatorname{div}\bar{\mathbf{V}}. \quad (11)$$

If we designate $\overline{b_0'^2} = -2\gamma$ and $\overline{b_0' b_1'} = \gamma$, then for an incompressible liquid Eq. (11) takes the form

$$\frac{\partial\bar{\mathbf{V}}}{\partial t}+(1-\gamma)\left(\operatorname{grad}\frac{\bar{\mathbf{V}}^2}{2}+\operatorname{rot}\bar{\mathbf{V}}\times\bar{\mathbf{V}}\right)=-\frac{1}{\rho}\operatorname{grad}\bar{p}+\nu\nabla^2\bar{\mathbf{V}}. \quad (12)$$

When $\gamma = 0$, (12) coincides in form with the Navier–Stokes equations. In the general case the parameter $\gamma$ can broaden the class of solutions of these equations.

Let us assume that the circular velocity v depends on the radius. Then in the steady-state case, according to (12), we will have the following dimensionless equation for it:

$$(1-\gamma)\left(u\frac{\partial v}{\partial r}+\frac{uv}{r}\right)=\frac{1}{\mathrm{Re}}\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial(vr)}{\partial r}\right]. \quad (13)$$

Let $v = 1/r^n$ and $u = -1/r$; then (13) changes into the algebraic equation

$$(1-\gamma)(n-1)=\frac{(n-1)(n+1)}{\mathrm{Re}}. \quad (14)$$

For the Navier–Stokes equations, Eq. (9) is converted into an identity only for n = 1. Consequently, these equations do not contain gradient solutions resulting in a change in the sign of $\omega_z$.

If $\gamma \neq 0$, then for any $n > 1$ Eq. (9) is converted into an identity through the choice of $\gamma$ from the equation $\gamma = 1 - \frac{n+1}{\mathrm{Re}}$.

## NOTATION

L, tornado length; $\omega$, motor speed; R, disk radius; g, free-fall acceleration; r, present radius; p, pressure; z, vertical coordinate; v, circular velocity component; $\rho$, density; $\bar{\mathbf{V}}$, average-velocity vector; $\bar{v}_j$, component of average velocity; $\nu$, kinematic viscosity; $x_{ok}$, coordinate of the center of gravity of a physically small volume; u, radial component of average velocity; Re, Reynolds number; U, voltage on motor terminals.

## LITERATURE CITED


1. A. H. Glaser, "The structure of a tornado vortex based on observational data," in: Conference on Cumulus Convection, C. E. Anderson, ed., Pergamon Press, New York (1959), pp. 207-229.
2. A. V. Lykov (A. W. Lykow) and V. A. Bubnov (W. A. Bubnow), "Über mögliche verallgemeinerungen der hydrodynamischen Gleichungen," Z. Angew. Math. Mech., 53, 281-288 (1973).